THERMAL STRESSES IN TUBES, PRODUCED FROM A MELT BY THE STEPANOV METHOD, DURING THEIR COOLING

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We calculated the temperature fields and the corresponding thermoelastic stresses in tubes, produced from a melt by the Stepanov method, during their cooling. The results of the calculations are presented in the form of surfaces constructed above the longitudinal cross section of the tube. We investigate the maximum values of stresses as a function of the rate of cooling and the behavior of the difference between the temperatures of the ambient inside and outside the tube.

Introduction. A number of publications, e.g., [1–3], present calculations of the temperature fields and corresponding thermal stresses in tubes, obtained from a melt by the Stepanov technique, during the growth of these tubes. In the present work we suggest calculations of thermoelastic stresses in tubes during their cooling, i.e., stresses varying in time. We also follow the tendency of the variation of these stresses as a function of such important parameters as the differences in the temperatures of the media inside and outside the tube. To determine the temperature field $T(r, z, t)$, varying in time, in a cooling-off crystal, it is necessary to know the initial distribution of the temperature $T^0(r, z)$. It was found in [1] and is used here.

For clarity we will show the distribution of the normal meridian σ_m and normal circumferential σ_φ stresses in the form of surfaces constructed above the longitudinal cross section of the tube at different instants of cooling. The calculations obtained permit one to obtain the initial data for optimizing the cooling process, primarily associated with the behavior of stresses in tubes and with the time of this process.

1. Mathematical Model. During the crystallization of a tube of length L, with the inner radius R_1 and the outer radius R_2, produced with the speed of pulling V_0, the temperature field in it T^0 satisfies the quasistationary equation of heat transfer

$$k_s \left(\frac{1}{r} \left(\frac{\partial}{\partial r} \left(r \frac{\partial T^0}{\partial r} \right) \right) + \frac{\partial^2 T^0}{\partial z^2} \right) - V_0 \rho_s c_{ps} \frac{\partial T^0}{\partial z} = 0 \tag{1}$$

subject to the following boundary conditions: on the inner and outer surfaces of the tube we prescribe heat exchange with the surrounding medium having the temperatures Θ_1 and Θ_2 (see Fig. 1):

$$- k_s \frac{\partial T^0}{\partial r} = h_s (T^0 - \Theta_2^0) |_{r=R_2}, \quad k_s \frac{\partial T^0}{\partial r} = h_s (T^0 - \Theta_1^0) |_{r=R_1}, \tag{2}$$

where

$$\Theta_1^0 = T_1^0 + \frac{z}{l} (T_2^0 - T_1^0); \quad \Theta_2^0 = T_3^0 + \frac{z}{l} (T_4^0 - T_3^0).$$

On the lower (the front of crystallization) and upper ends of the tube the following temperatures are prescribes:

$$T^0 (r, 0) = T_m^0, \quad T^0 (r, l) = T_c^0, \quad R_1 \le r \le R_2. \tag{3}$$

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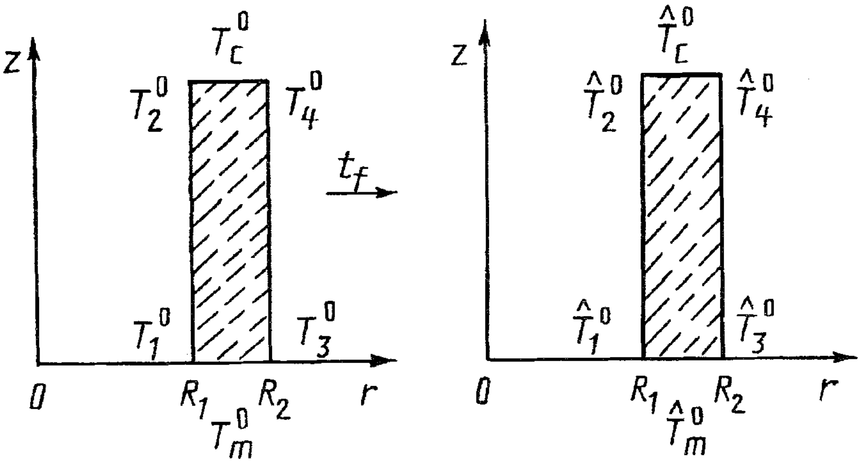

Fig. 1. Scheme of a heat node at the initial and final instants of time.

The temperature $T(r, z, \tau)$ in a cooling-off crystal satisfies the following heat conduction equation:

$$\frac{\partial T}{\partial r} = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right) + \frac{\partial^2 T}{\partial z^2}, \quad \tau = at, \quad a = \frac{k_s}{c_{ps}\rho_s}, \tag{4}$$

with the boundary conditions

$$-k_s\frac{\partial T}{\partial r} = h_s(T - \Theta_2)|_{r=R_2}, \quad k_s\frac{\partial T}{\partial r} = h_s(T - \Theta_1)|_{r=R_1},$$

$$T(r, 0, \tau) = T_m, \quad T(r, l, \tau) = T_c, \quad R_1 \le r \le R_2, \tag{5}$$

and the initial condition

$$T(r, z, 0) = T^0(r, z). \tag{6}$$

It is assumed that the temperatures Θ_1, Θ_2 and T_m, T_c vary in time according to the exponential law

$$\Theta_1 = T_1^0 e^{-\alpha_1 t} + \frac{z}{l}(T_1^0 e^{-\alpha_2 t} - T_1^0 e^{-\alpha_1 t}),$$

$$\Theta_2 = T_3^0 e^{-\alpha_3 t} + \frac{z}{l}(T_4^0 e^{-\alpha_4 t} - T_3^0 e^{-\alpha_3 t}), \quad T_m = T_m^0 e^{-\alpha_5 t}, \quad T_c = T_c^0 e^{-\alpha_6 t}. \tag{7}$$

We determine the coefficients α_1, α_2, β_1, β_2, γ_1, and γ_2 from the condition that we know the final thermal state of the entire system, at which it arrives after a certain known time t_f.

We present the solution of problem (4)-(6) in the form of the sum

$$T = T_1 + T^*, \tag{8}$$

where

$$T^* = \lambda\frac{\Theta_1 - \Theta_2}{w_1 + w_2}\ln r + \frac{\Theta_1 w_2 + \Theta_2 w_1}{w_1 + w_2},$$

$$w_1 = \frac{1}{R_1} - \lambda\ln R_1, \quad w_2 = \frac{1}{R_2} + \lambda\ln R_2, \quad \lambda = h_s/k_s. \tag{9}$$

Let us substitute Eq. (8) into Eqs. (4)-(6) and then apply a Laplace transformation to T_1. Then for $\widetilde{T}_1$, which is the Laplace transform of the function T_1, we obtain the following problem:

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial\widetilde{T}_1}{\partial r}\right)\right)+\frac{\partial^2\widetilde{T}_1}{\partial z^2}-s\widetilde{T}_1 \overset{\text{def}}{=} F(r,z,s),\quad F(r,z,s)=s\widetilde{T}^*-T^0,$$

$$-k_s\frac{\partial\widetilde{T}_1}{\partial r}=h_s\widetilde{T}_1|_{r=R_2},\quad k_s\frac{\partial\widetilde{T}_1}{\partial r}=h_s\widetilde{T}_1|_{r=R_1},\tag{10}$$

$$\widetilde{T}_1(r,0,s)=\widetilde{T}_m-\widetilde{T}^*(r,0),\quad \widetilde{T}_1(r,l,s)=\widetilde{T}_c-\widetilde{T}^*(r,l).$$

The resulting problem (10) admits separation of variables, and its solution can be represented by the formula

$$\widetilde{T}_1(r,z,s)=\sum_{k=1}^{\infty}\widetilde{Z}_k(z,s)X_k(r).\tag{11}$$

Here the functions $X_k(r)$ are defined by the equalities

$$X_k=\frac{D_k(r)}{\|D_k\|},\quad D_k=D\left(\frac{\mu_k}{R_2}r\right)=J_0\left(\frac{\mu_k}{R_2}r\right)+\gamma(\mu_k)N_0\left(\frac{\mu_k}{R_2}r\right),$$

where J_0, N_0 are Bessel functions of zero order of the first and second kind, respectively. The eigenvalues μ_k are solutions of the algebraic equation

$$\begin{vmatrix}\mu J_1(\mu)-kJ_0(\mu), & \mu N_1(\mu)-kN_0(\mu)\\ \mu J_1\left(\mu\frac{R_1}{R_2}\right)+kJ_0\left(\mu\frac{R_1}{R_2}\right), & \mu N_1\left(\mu\frac{R_1}{R_2}\right)+kN_0\left(\mu\frac{R_1}{R_2}\right)\end{vmatrix}=0,$$

and the numbers $\gamma(\mu_k)$ are equal to

$$\gamma(\mu_k)=-\frac{\mu_kJ_1(\mu_k)-kJ_0(\mu_k)}{\mu_kN_1(\mu_k)-kN_0(\mu_k)}.$$

The square of the norm of the functions D_k is

$$\|D_k\|^2=\frac{(k^2+\mu_k^2)}{2\lambda_k}\left[D^2(\mu_k)-\left(\frac{R_1}{R_2}\right)^2D^2\left(\mu_k\frac{R_1}{R_2}\right)\right],\quad \lambda_k=(\mu_k/R_2)^2.$$

The functions $\widetilde{Z}_k(z,s)$ have the form

$$\widetilde{Z}_k(z,s)=a_k\frac{\operatorname{Sh}\delta_k(l-z)}{\operatorname{Sh}\delta_kl}+b_k\frac{\operatorname{Sh}\delta_kl}{\operatorname{Sh}\delta_kl}-\int_0^1 G_k(z,y)F_k(y)\,dy,\tag{12}$$

where

$$G_k(z,y)=\frac{1}{\delta_k\operatorname{Sh}\delta_kl}\begin{cases}\operatorname{Sh}\delta_kz\operatorname{Sh}\delta_k(l-y), & 0\le z\le y,\\ \operatorname{Sh}\delta_k(l-z)\operatorname{Sh}\delta_ky, & y\le z\le l;\end{cases}$$

$$\delta_k=\sqrt{s+\lambda_k};\quad a_k=(\widetilde{T}_m-\widetilde{T}^*(0,s),\ X_k)_r=\frac{T_m^0}{s+\frac{\alpha_5}{a}}m_k+\frac{T_1^0}{s+\frac{\alpha_1}{a}}A_k^{(1)}-$$

$$-\frac{T_3^0}{s+\frac{\alpha_3}{a}}A_k^{(2)};\quad b_k=(\widetilde{T}_c-\widetilde{T}^*(l,\,s),\;X_k)_r=\frac{T_c^0}{s+\frac{\alpha_6}{a}}m_k+$$

$$+\frac{T_2^0}{s+\frac{\alpha_2}{a}}A_k^{(1)}-\frac{T_4^0}{s+\frac{\alpha_4}{a}}A_k^{(2)};\quad A_k^{(1)}=\frac{\lambda l_k-w_2m_k}{w_1+w_2};$$

$$A_k^{(2)}=\frac{\lambda l_k-w_1m_k}{w_1+w_2};\quad m_k=\frac{k}{\lambda_k\,\|D_k\|}\left[D(\mu_k)+\frac{R_1}{R_2}D\left(\mu_k\frac{R_1}{R_2}\right)\right];$$

$$l_k=\frac{1}{\lambda_k\,\|D_k\|}\left[kD(\mu_k)\ln R_2+k\frac{R_1}{R_2}D\left(\mu_k\frac{R_1}{R_2}\right)\ln R_1+D(\mu_k)-D\left(\mu_k\frac{R_1}{R_2}\right)\right];$$

$$F_k=(F,\,X_k)_r=-Z_k^0-(T_0^*,\,X_k)_r+s(\widetilde{T}^*,\,X_k)_r.$$

We denote by $(\cdot,\,\cdot)_r$ the scalar product with the weight r in the space $L_2^r(0,\,l)$. In the expression for F_k the functions Z_k^0 and $(T_0^*,\,X_k)_r$ are the coefficients of the Fourier expansion of T^0 in the functions X_k:

$$T^0=\sum_{k=1}^{\infty}(T_0^*,\,X_k)_rX_k+\sum_{k=1}^{\infty}Z_k^0X_k,\quad T_0^*=T^*(r,\,z,\,0).$$

Let us present explicit expressions for the functions Z_k^0, $(T_0^*, X_k)_r$, $s(\widetilde{T}^*, X_k)_r$ entering into the representation for F_k. The functions Z_k^0 have the form

$$Z_k^0=\exp\left(\frac{\chi z}{2}\right)\left[G_k^{(1)}\exp\left(-\frac{\eta_kz}{2}\right)+G_k^{(2)}\exp\left(-\frac{\eta_kz}{2}\right)\right]-\frac{\overset{0}{c}_k}{\lambda_k},$$

where

$$G_k^{(1)}=\frac{\left(\overset{0}{b}_k+\frac{\overset{0}{c}_k}{\lambda_k}\right)\exp\left(-\frac{l(\eta_k+\chi)}{2}\right)-\left(\overset{0}{a}_k+\frac{\overset{0}{c}_k}{\lambda_k}\right)}{\exp(-\eta_kl)-1};$$

$$G_k^{(2)}=\frac{\left(\overset{0}{b}_k+\frac{\overset{0}{c}_k}{\lambda_k}\right)\exp\left(\frac{l(\eta_k-\chi)}{2}\right)-\left(\overset{0}{a}_k+\frac{\overset{0}{c}_k}{\lambda_k}\right)}{\exp(-\eta_kl)-1};$$

$$\chi=\frac{V_0\rho_sc_{ps}}{k_s};\quad \eta_k=\sqrt{\chi^2+4\lambda_k};\quad \overset{0}{a}_k=\delta_1m_k+\gamma_1l_k.$$

The coefficients $\overset{0}{b}_k$ are calculated from the same formulas as $\overset{0}{a}_k$, except that it is necessary to take δ_2, γ_2 instead of δ_1, γ_1. The coefficients δ_i, γ_i $(i=1,\,2)$ are defined by the following expressions:

$$\delta_1=T_m^0-\frac{T_1^0w_2+T_3^0w_1}{w_1+w_2},\quad \gamma_1=-\lambda\frac{T_3^0-T_1^0}{w_1+w_2},$$

$$\delta_2=T_c^0-\frac{T_2^0w_2+T_4^0w_1}{w_1+w_2},\quad \gamma_2=-\lambda\frac{T_4^0-T_2^0}{w_1+w_2}.$$

The coefficients c_k^0 can be found from the formulas

$$c_k^0 = \frac{\chi}{\lambda_k \| D_k \|} \left\{ [\alpha (k \ln R_2 + 1) + \beta k] D(\mu_k) + \right.$$

$$\left. + \left[\alpha \left(k \frac{R_1}{R_2} \ln R_1 - 1 \right) + \beta \frac{R_1}{R_2} k \right] D \left(\mu_k \frac{R_1}{R_2} \right) \right\}.$$

In the latter expressions

$$\alpha = \lambda \frac{T_1^0 - T_2^0 - T_3^0 + T_4^0}{l (w_1 + w_2)}, \quad \beta = \frac{(T_2^0 - T_1^0) w_2 + (T_4^0 - T_3^0) w_1}{l (w_1 + w_2)}.$$

The coefficients $(T_0^*, X_k)_r$ can be written as follows:

$$(T_0^*, X_k)_r = P_k^{(1)} + \frac{z}{l} P_k^{(2)},$$

where

$$P_k^{(1)} = \lambda l_k \frac{T_3^0 - T_1^0}{w_1 + w_2} + m_k \frac{T_1^0 w_2 + T_3^0 w_1}{w_1 + w_2}, \quad P_k^{(2)} = l (\alpha l_k + \beta m_k).$$

And, finally, the coefficients $s(\widetilde{T}^*, X_k)_r$ can be found from the formula

$$s(\widetilde{T}^*, X_k)_r = s(V_k^{(1)}(s) + z V_k^{(2)}(s)),$$

where

$$V_k^{(1)}(s) = -\frac{T_1^0}{s + \dfrac{\alpha_1}{a}} A_k^{(1)} + \frac{T_3^0}{s + \dfrac{\alpha_3}{a}} A_k^{(2)};$$

$$V_k^{(2)}(s) = -\frac{A_k^{(1)}}{l} \left(\frac{T_2^0}{s + \dfrac{\alpha_2}{a}} - \frac{T_1^0}{s + \dfrac{\alpha_1}{a}} \right) +$$

$$+ \frac{A_k^{(2)}}{l} \left(\frac{T_4^0}{s + \dfrac{\alpha_4}{a}} - \frac{T_3^0}{s + \dfrac{\alpha_3}{a}} \right).$$

Substituting the above expressions into Eq. (11) and integrating, we find

$$\int_0^l G_k(z, y) F_k(y)\, dy = \frac{G_k^{(1)}}{\left(\dfrac{\chi - \eta_k}{2} \right)^2 - \delta_k^2} \times$$

$$\times \left[\frac{\mathrm{Sh}\, \delta_k (l - z) + \exp \left(\dfrac{\lambda - \eta_k}{2} l \right) \mathrm{Sh}\, \delta_k z}{\mathrm{Sh}\, \delta_k l} - \exp \left(\frac{\chi - \eta_k}{2} z \right) \right] +$$

$$+\frac{G_k^{(2)}}{\left(\frac{\chi+\eta_k}{2}\right)^2-\delta_k^2}\left[\frac{\operatorname{Sh}\delta_k(l-z)+\exp\left(\frac{\chi+\eta_k}{2}l\right)\operatorname{Sh}\delta_k z}{\operatorname{Sh}\delta_k l}-\right.$$

$$\left.-\exp\left(\frac{\chi+\eta_k}{2}z\right)\right]-\frac{c_k^0}{\lambda_k\delta_k^2}\left[1-\frac{\operatorname{Sh}\delta_k z+\operatorname{Sh}\delta_k(l-z)}{\operatorname{Sh}\delta_k l}\right]+$$

$$+\frac{P_k^{(1)}}{\delta_k^2}\left[1-\frac{\operatorname{Sh}\delta_k z+\operatorname{Sh}\delta_k(l-z)}{\operatorname{Sh}\delta_k l}\right]+\frac{P_k^{(2)}}{\delta_k^2}\left[z-l\frac{\operatorname{Sh}\delta_k z}{\operatorname{Sh}\delta_k l}\right]-$$

$$-\frac{sV_k^{(1)}(s)}{\delta_k^2}\left[1-\frac{\operatorname{Sh}\delta_k z+\operatorname{Sh}\delta_k(l-z)}{\operatorname{Sh}\delta_k l}\right]-\frac{sV_k^{(2)}(s)}{\delta_k^2}\left[z-l\frac{\operatorname{Sh}\delta_k z}{\operatorname{Sh}\delta_k l}\right]. \tag{13}$$

Using the well-known inversion theorems, applied to the functions $\widetilde{Z}_k(z, s)$ of formulas (12), (13), we determine the temperature $T_1(z, t)$:

$$T_1(r, z, t)=\sum_{k=1}^{\infty} Z_k(z, t)X_k(r).$$

Here

$$Z_k(z, t)=\sum_{s=1}^{2}\frac{\alpha_{2s}}{a}\frac{(-1)^j U_k^{(2s)}}{\lambda_k-\frac{\alpha_{2s}}{a}}\exp(-\alpha_{2s}t)+$$

$$+2\frac{c_k^0}{\lambda_k}\sum_{j=1}^{\infty}(-1)^j\frac{W_j\sin\frac{j\pi z}{l}}{j\pi}\exp\left(-\left(\frac{j^2\pi^2}{l^2}+\lambda_k\right)at\right)+$$

$$+2\sum_{s=1}^{4}\sum_{j=1}^{\infty}(-1)^j C_k^{(s)}\frac{\left(\frac{j^2\pi^2}{l^2}+\lambda_k\right)\sin\frac{j\pi z}{l}}{j\pi\left(\frac{j^2\pi^2}{l^2}+\lambda_k-\frac{\alpha_s}{a}\right)}\exp\left(-\left(\frac{j^2\pi^2}{l^2}+\lambda_k\right)at\right)+$$

$$+2\pi\sum_{s=1}^{2}\sum_{j=1}^{\infty}jC_k^{(s)}\frac{\sin\frac{j\pi z}{l}\left[1+(-1)^{j+1}\exp\left(\frac{\chi-(-1)^s\eta_k}{2}l\right)\right]}{l^2\left(\frac{j^2\pi^2}{l^2}+\left(\frac{\chi-(-1)^s\eta_k}{2}\right)^2\right)}\times$$

$$\times\exp\left(-\left(\frac{j^2\pi^2}{l^2}+\lambda_k\right)at\right)-\sum_{s=1}^{3}\frac{H_k^{(s)}\operatorname{Sh}\sqrt{\left(\lambda_k-\frac{\alpha_{2s-1}}{a}\right)}(l-z)}{\operatorname{Sh}\sqrt{\left(\lambda_k-\frac{\alpha_{2s-1}}{a}\right)}}\times$$

$$\times \exp(-\alpha_{2s-1} t) + \sum_{s=1}^{3} \frac{B_k^{(s)} \operatorname{Sh} \sqrt{\left(\lambda_k - \frac{\alpha_{2s}}{a}\right)}\, z}{\operatorname{Sh} \sqrt{\left(\lambda_k - \frac{\alpha_{2s}}{a}\right)}} \exp(-\alpha_{2s} t) +$$

$$+ 2\pi \sum_{s=1}^{6} \sum_{j=1}^{\infty} jU_k^{(s)} \frac{\sin \frac{j\pi z}{l}}{l^2 \left(\frac{j^2\pi^2}{l^2} + \lambda_k - \frac{\alpha_s}{a}\right)} \exp\left(-\left(\frac{j^2\pi^2}{l^2} + \lambda_k\right) at\right) -$$

$$- 2 \sum_{j=1}^{\infty} (-1)^j \frac{[P_k^{(1)} W_j + P_k^{(2)}] \sin \frac{j\pi z}{l}}{j\pi} \exp\left(-\left(\frac{j^2\pi^2}{l^2} + \lambda_k\right) at\right).$$

In this expression the following notation is introduced

$$U_k^{(1)} = -T_1^0 A_k^{(1)}, \quad U_k^{(2)} = (-1)^i T_2^0 A_k^{(1)}, \quad U_k^{(3)} = -T_3^0 A_k^{(2)},$$

$$U_k^{(4)} = (-1)^{j+1} T_4^0 A_k^{(2)}, \quad U_k^{(5)} = -T_m^0 m_k, \quad U_k^{(6)} = (-1)^j T_c^0 m_k,$$

$$H_k^{(1)} = U_k^{(1)} \left(1 - \frac{\alpha_1}{a}\right), \quad H_k^{(2)} = U_k^{(2)} \left(1 - \frac{\alpha_3}{a}\right), \quad H_k^{(3)} = U_k^{(3)},$$

$$B_k^{(1)} = (-1)^j U_k^{(2)} \left(1 - \frac{\alpha_2}{a}\right), \quad B_k^{(2)} = (-1)^j U_k^{(4)} \left(1 - \frac{\alpha_4}{a}\right),$$

$$B_k^{(3)} = (-1)^j U_k^{(6)}, \quad C_k^{(1)} = U_k^{(1)} (1 - W_j), \quad C_k^{(2)} = U_k^{(2)} (-1)^j,$$

$$C_k^{(3)} = U_k^{(3)} (1 - W_j), \quad C_k^{(4)} = U_k^{(4)} (-1)^j, \quad W_j = \begin{cases} 2, & j = 2m + 1, \\ 0, & j = 2m. \end{cases}$$

Thus, we found the function $T_1(r, z, t)$ and this means that we found the unknown temperature field $T(r, z, t)$. The computed temperature field makes it possible to determine the corresponding thermoelastic stressed state. For this purpose, we will represent a tube in the form of a circular cylindrical shell of constant thickness. Let us denote the thickness of the shell by $h = R_2 - R_1$, the radius of the middle surface by $R = (R_1 + R_2)/2$, and the axial and radial displacements of the middle surface by u and ω, respectively; σ_m and σ_φ are the meridian and circumferential normal stresses.

The thermal stresses σ_m and σ_φ are determined from the well-known formulas [4]

$$\sigma_\varphi = \frac{E}{1 - \nu^2} \left[\nu \frac{du}{dz} - \nu x \frac{d^2\omega}{dz^2} + \frac{\omega}{R} - (1 + \nu) \alpha_t T\right],$$

$$\sigma_m = \frac{E}{1 - \nu^2} \left[\frac{du}{dz} - x \frac{d^2\omega}{dz^2} + \nu \frac{\omega}{R} - (1 + \nu) \alpha_t T\right], \tag{14}$$

$$u = \int \left[(1 + \nu) \overline{T} - \nu \frac{\omega}{R}\right] dz, \quad M = -D \left[\frac{d^2\omega}{dz^2} + (1 + \nu) \alpha_t \overline{T}\right].$$

The value of x in formulas (14) is reckoned from the middle surface of the tube $(-0.5h < x < 0.5h)$.
The component of the displacement vector ω satisfies the equation

$$\frac{d^4\omega}{dz^4} + 4k^4\omega = \frac{E h \alpha_t}{DR}\bar{T} - (1+\nu)\,\alpha_t \frac{d^2\bar{\bar{T}}}{dz^2} = f(z)\,, \tag{15}$$

$$\bar{T} = \frac{1}{h}\int_{-h/2}^{h/2} T(R+x,\ z)\,dx\,,\quad \bar{\bar{T}} = \frac{12}{h^3}\int_{-h/2}^{h/2} T(R+x,\ z)\,x\,dx\,.$$

The boundary conditions for Eq. (15) are formulated for a shell with free edges, i.e.,

$$M = \frac{dM}{dz} = 0\,,\quad z = 0\,,\ l\,. \tag{16}$$

The general solution of Eq. (15) can be written in the form

$$\omega(z) = C_1 \operatorname{sh} kz \cos kz + C_2 \frac{1}{2}\left[\operatorname{ch} kz \sin kz + \operatorname{sh} kz \cos kz\right] +$$

$$+ C_3 \frac{1}{2} \operatorname{sh} kz \sin kz + C_4 \frac{1}{4}\left[\operatorname{ch} kz \sin kz - \operatorname{sh} kz \cos kz\right] +$$

$$+ \frac{1}{4k^3}\int_0^z \left[\operatorname{ch} k(z-\xi)\sin(z-\xi) - \operatorname{sh}(z-\xi)\cos(z-\xi)\right] f(\xi)\,d\xi\,.$$

The coefficients C_1, C_2, C_3, C_4 can be found from boundary conditions (16). In the preceding formulas the quantities k and D denote

$$k^4 = \frac{3(1-\nu^2)}{h^2R^2}\,,\quad D = \frac{E h^3}{12(1-\nu^2)}\,.$$

The characteristics of the materials and the parameters of the process are taken from handbooks [5, 6]. It should be noted that the heat transfer coefficient we selected takes into account the total amount of heat removed by radiation and convection.

2. Numerical Results. In order to calculate the thermal stresses in a cooling-off crystal, it is necessary first to calculate the coefficients α_1, α_2, β_1, β_2, γ_1, and γ_2, which determine the rate of cooling. They can be found from the following equalities (see Fig. 1):

$$T_1^0 e^{-\alpha_1 t_f} = \hat{T}_1^0\,,\quad T_3^0 e^{-\alpha_2 t_f} = \hat{T}_3^0\,,\quad T_2^0 e^{-\alpha_3 t_f} = \hat{T}_2^0\,,$$

$$T_4^0 e^{-\alpha_4 t_f} = \hat{T}_4^0\,,\quad T_m^0 e^{-\alpha_5 t_f} = \hat{T}_m^0\,,\quad T_c^0 e^{-\alpha_6 t_f} = \hat{T}_c^0\,.$$

Here $\hat{T}_1^0$, $\hat{T}_2^0$, $\hat{T}_3^0$, $\hat{T}_4^0$ are the final values of the temperatures of the ambient and $\hat{T}_m^0$, $\hat{T}_c^0$ are the final values of the temperatures of the lower and upper ends of the crystal, respectively. The system arrives at this temperature state after a certain known time $t = t_f$. Figures 2 and 3 present a typical distribution of the stresses σ_φ and σ_m for different times and the following values of the parameters that characterize the thermal zone:

$$T_1^0 = 2000^\circ\,,\quad T_2^0 = 1540^\circ\,,\quad T_3^0 = 1875^\circ\,,\quad T_4^0 = 1520^\circ\,,\quad T_m^0 = 2050^\circ\,,\quad T_c^0 = 1550^\circ\,,$$

$$\hat{T}_1^0 = 200^\circ\,,\quad \hat{T}_2^0 = 180^\circ\,,\quad \hat{T}_3^0 = 160^\circ\,,\quad \hat{T}_4^0 = 150^\circ\,,\quad \hat{T}_m^0 = 275^\circ\,,\quad \hat{T}_c^0 = 200^\circ\,\mathrm{C}\,. \tag{17}$$

According to our calculations, the maximum values $|\sigma_\varphi|_{\max}$ are attained at the lower end of the tube.

The figures also show a smooth change in the stresses σ_φ and σ_m with retention of all the characteristic features of the initial state.

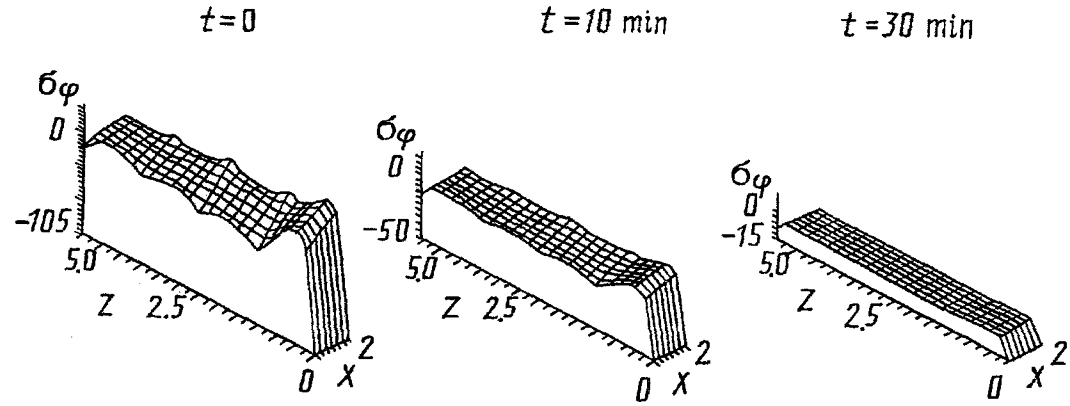

Fig. 2. Typical distribution of stresses σ_φ at different instants of cooling: σ_φ, MPa; Z, cm; X, mm.

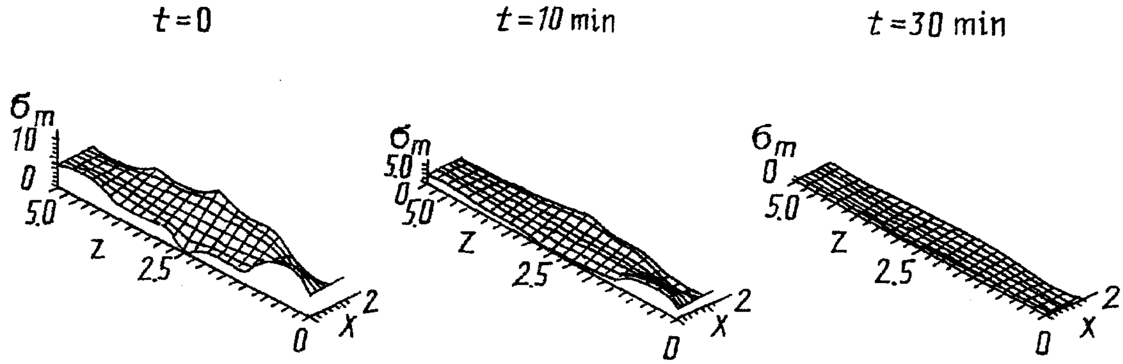

Fig. 3. Typical distribution of stresses σ_m at different instants of cooling. σ_m, MPa.

Of course, the character of the behavior of the stresses depends on the rate of cooling, i.e., on the values of the coefficients α_1, α_2, β_1, β_2, γ_1, γ_2 and, which is very interesting for the practical growth of crystals, on the difference between the temperatures of the ambient inside and outside the surface of the tube.

We constructed the dependences of the maximum values of the stresses $|\sigma_\varphi|_{\max}$ for the process of tube cooling over the periods of time $t_f = 60$ and 120 min for different characters of the behavior of the difference between the temperatures of the external media: for a decreasing (Fig. 4a) and an increasing (Fig. 4b) one. Figure 4a corresponds to the temperature regime (17) and Fig. 4b to the temperature regime

$$T_1^0 = 2000^\circ, \quad T_2^0 = 1540^\circ, \quad T_3^0 = 1980^\circ, \quad T_4^0 = 1520^\circ, \quad T_m^0 = 2050^\circ, \quad T_c^0 = 1550^\circ,$$

$$\hat{T}_1^0 = 280^\circ, \quad \hat{T}_2^0 = 190^\circ, \quad \hat{T}_3^0 = 160^\circ, \quad \hat{T}_4^0 = 150^\circ, \quad \hat{T}_m^0 = 300^\circ, \quad \hat{T}_c^0 = 200^\circ\mathrm{C}. \tag{18}$$

From these graphs it is seen that the change in the stresses is very appreciable on the lower end of the tube in contrast to the upper end. Thus, for example, the maximum value $|\sigma_\varphi|_{\max}$ on the lower end decreased from 105 to 20 MPa and on the upper end from 17 to 7 MPa. We note that the behavior of $|\sigma_\varphi|_{\max}$ depends qualitatively on the behavior of the temperature differences: while $|\sigma_\varphi|_{\max}$ decreases monotonically with time with a decreasing temperature difference (see Fig. 4a), in the case of an increasing difference it increases substantially up to a certain instant of time and then decreases, exhibiting a pronounced maximum (see Fig. 4b). Our calculations show that the largest values of $|\sigma_\varphi|_{\max}$ in the case of an increasing temperature difference are almost independent of the rate of cooling and are attained at different instants of time t^*. For example, for the cooling rate corresponding to t_f = 60 min t^* = 15 min, and for t_f = 120 min t^* = 40 min (see Fig. 4b).

The maximum values of $|\sigma_m|_{\max}$ are attained on the tube surface near the lower end. As a whole the character of the behavior of $|\sigma_m|_{\max}$ in time is similar to the behavior of $|\sigma_\varphi|_{\max}$ and differs only in the values of the stresses, which, as a rule, are smaller by almost an order of magnitude.

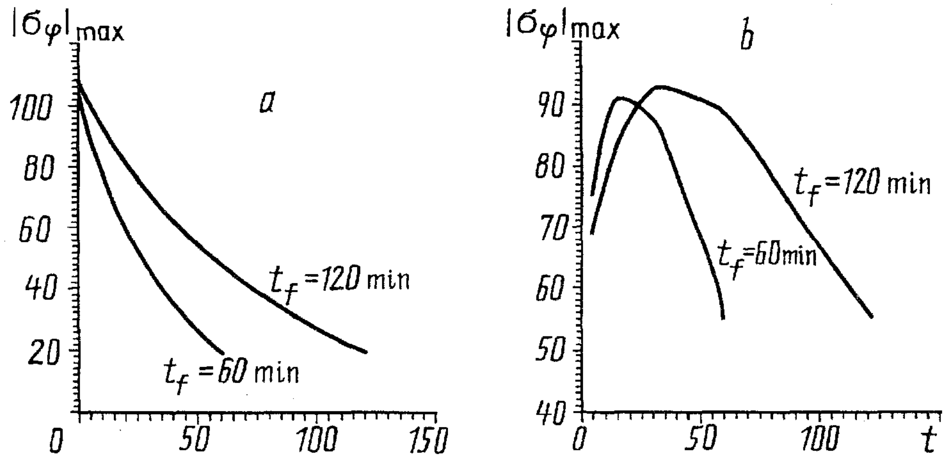

Fig. 4. Time dependences of $|\sigma_\varphi|_{max}$ for different rates of cooling: a) an increasing difference of temperatures; b) a decreasing difference of temperatures. $|\sigma_\varphi|_{max}$, MPa; t, min.

CONCLUSIONS

1. The stresses in the process of the cooling-off of a crystal preserve all of the characteristic features of the initial state, changing smoothly with time.

2. The behavior of the maximum stresses $|\sigma_\varphi|_{max}$ in time may either be monotonically decreasing or have a maximum during the time of cooling t_f, depending on the regime of the crystal cooling-off.

NOTATION

k_s, thermal conductivity coefficient; V_0, speed of crystal pulling; ρ_s, density of the crystal; T_m^0, melting temperature; T_c^0, temperature of the upper end of the crystal; α_i, coefficient of temperature expansion; h_s, heat transfer coefficient; c_{ps}, heat capacity; E, Young's modulus; ν, Poisson coefficient; σ_φ, σ_m, circumferential and meridian normal stresses; t_f, time of crystal cooling-off.

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